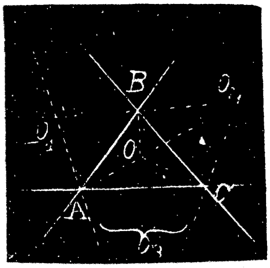
II. Solution by **WALTER H. DRANE**, Graduate Student at Harvard University, and **J. SCHEFFER**, A. M., Hagerstown, Md.

Let ABC be the given triangle, O, O_1, O_2, O_3, the centers of the inscribed and the three escribed circles. Then O_1BO_2, O_1AO_3, O_2CO_3, and AOO_2 are straight lines; also OB, OA, OC are each perpendicular to O_1BO_2, O_1AO_3, and O_2CO_3, respectively. In triangles AOC and BAO_2, $\angle OAC = BAO_2$ and $\angle OCA = \angle BO_2A$ since we have $\angle OCA = 90^\circ - \angle ACO_3 = 90^\circ - \frac{1}{2}(\angle CAB + \angle CBA) = 90^\circ - (\angle OAB + OBA) = 90^\circ - [180^\circ - (\angle O_1AB + \angle O_1BA)] = 90^\circ - \angle O = \angle O_1O_2A$.



$\because$ triangles AOC and ABO_2 are similar, and we have

$$AO : AB :: AC : AO_2 \quad \ldots\ldots\ldots(1).$$

Again in triangles O_1BA and AO_3C, $\angle O = \angle ACO_3$ and $\angle O_3AC = \angle O_1AB$. Hence triangles O_1BA and AO_3C are similar, and we have,

$$AO_1 : AC :: AB : AO_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Multiplying (1) by (2) $AO.AO_1 : AB.BC : AB.AC : AO_2.AO_3$.

$\because AO.AO_1.AO_2.AO_3 = AB^2.AC^2$. Q. E. D.

III. Solution by **G. B. M. ZERR**, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In the figure of the last solution, draw the lines OD and O_1D_1 perpendicular to AC. Then $AO = \surd(OD^2 + AD^2) = \surd(r^2 + r^2\cot^2 \frac{1}{2}A) = r\operatorname{cosec}\frac{1}{2}A$.

Similarly, $AO_1 = r_1\operatorname{cosec}\frac{1}{2}A$.

$AO_2 = \surd(OO_2^2 - AO^2) = AO\surd[(OO_2^2/AO^2) - 1] = AO\cot\frac{1}{2}C$.

Similarly, $AO_3 = AO\cot\frac{1}{2}B$.

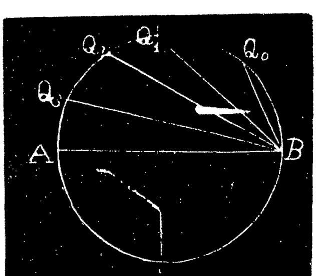
$$\begin{aligned}\therefore AO.AO_1.AO_2.AO_3 &= r^3r_1\operatorname{cosec}^4 \tfrac{1}{2}A \cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\\ &= (s-a)(s-b)(s-c)s\operatorname{cosec}^4 \tfrac{1}{2}A \tan^2 \tfrac{1}{2}A\\ &= s(s-a)(s-b)(s-c)/\sin^2 \tfrac{1}{2}A \cos^2 \tfrac{1}{2}A\\ &= 4s(s-a)(s-b)(s-c)/\sin^2 A = 4S^2/\sin^2 A.\end{aligned}$$

But $\sin A = 2S/bc = 2S/AB.AC$. $\therefore AO.AO_1.AO_2.AO_3 = AB^2.AC^2$.

Also solved by *ELMER SCHUYLER*.

101. Proposed by **E. W. MORRELL**, A.M., Late Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.

AB is the diameter of a circle and Q_0 any point on the circumference; Q_1, Q_2, Q_3.... are the points of bisection of the arcs AQ_0, AQ_1, AQ_2.... Prove that BQ_1, BQ_2, BQ_3 .... $BQ_n = OA^n.(AQ_0/AQ_n)$.



Solution by **G. B. M. ZERR**, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; **J. SCHEFFER**, A.M., Hagerstown, Md., and **ELMER SCHUYLER**, High Bridge, N. J.

Let O be the center of the circle.

$\angle ABQ_0 = \theta$.

$\therefore BQ_1 = AB\cos\frac{1}{2}\theta$, $BQ_2 = AB\cos(\theta/2^2)$.

$BQ_3 = AB\cos(\theta/2^3)$, $BQ_n = AB\cos(\theta/2^n)$.

$$AQ_0=AB\sin\theta=2AB\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta=2^2AB\sin(\theta/2^2)\cos(\theta/2^2)\cos\tfrac{1}{2}\theta$$
$$=2^nAB\sin(\theta/2^n)\cos\tfrac{1}{2}\theta\cos(\theta/2^2)\ldots\cos(\theta/2^n).$$

$AQ_n=AB\sin(\theta/2^n)$.

$\therefore\ (AQ_0/AQ_n)=2^n\cos\frac{1}{2}\theta\cos(\theta/2^2)\ldots\cos(\theta/2^n)$.

$$BQ_1.BQ_2.BQ_3\ldots BQ_n=(AB^n)\cos\tfrac{1}{2}\theta\cos(\theta/2^2)\ldots\cos(\theta/2^n)$$
$$=(\tfrac{1}{2}AB)^n(AQ_0/AQ_n)=(AO)^n(AQ_0/AQ_n).$$

CALCULUS.

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78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Investigate value of $\left(\dfrac{\tan x}{x}\right)^{1/x^n}$ where x is 0 and n has consecutive values 1, 2, 3, 4, ... Is there any law governing the different results? When $n=1$, result is 1; when $n=2$, result is $e^{\frac{1}{3}}$; $n=3$, gives ∞, etc.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

$$\left(\frac{\tan x}{x}\right)^{1/x^n}=e^{(1/x^n)\log[(\tan x)/x]}=y.$$

$$\text{Limit of }\frac{\log\tan x-\log x}{x^n}=\text{limit of }\frac{\cot x\sec^2x-(1/x)}{nx^{n-1}}=\text{limit of }\frac{2x-\sin 2x}{nx^n\sin 2x},$$

$$\text{but }\sin 2x=2x-\frac{8x^3}{6}+\frac{32x^5}{120}-\frac{128x^7}{5040},\text{ etc.},=2x-\frac{4x^3}{3}+\frac{4x^5}{15}-\frac{4x^7}{45}+,\text{ etc.}$$

$$\therefore\ \frac{2x-\sin 2x}{nx^n\sin 2x}=\frac{2x-2x+\dfrac{4x^3}{3}-\dfrac{4x^5}{15}+\dfrac{4x^7}{45}-,\text{ etc.}}{nx^{n+1}\left(2-\dfrac{4x^2}{3}+\dfrac{4x^4}{15}-\dfrac{4x^6}{45}+,\text{ etc.}\right.}=\frac{30-6x^2+2x^4}{nx^{n-2}(45-30x^2+6x^4)},\text{ ap-}$$

$$\text{proximately, }=\frac{2}{3nx^{n-2}}+\frac{14}{45nx^{n-4}}+,\text{ etc.},=S.$$

When $n=1$, $S=0$ for $x=0$.
When $n=2$, $S=\frac{1}{3}$ for $x=0$.
When $n=3$, 4, 5, etc., $S=\infty$ for $x=0$.
$\therefore$ When $n=1$, $y=e^0=1$.
When $n=2$, $y=e^{\frac{1}{3}}$.
When $n=3$, 4, 5, etc., $y=e^\infty=\infty$.

Also solved by *ELMER SCHUYLER*, whose solution has been accidentally misplaced, and hence does not appear in this issue.